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Fundamental study

The multiple facets of the canonical direct unit implicational basis

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ABSTRACT

The notion of *dependencies* between "attributes" arises in many areas such as relational databases, data analysis, data-mining, formal concept analysis, knowledge structures . . .. Formalization of dependencies leads to the notion of so-called *full implicational systems* (or full family of functional dependencies) which is in one-to-one correspondence with the other significant notions of *closure operator* and of *closure system*. An efficient generation of a full implicational system (or a closure system) can be performed from equivalent implicational systems and in particular from the bases for such systems, for example, the so-called *canonical basis*. This paper shows the equality between five other bases originating from different works and satisfying various properties (in particular they are unit implicational systems). The three main properties of this unique basis are the directness, canonical and minimal properties, whence the name *canonical direct unit implicational basis* given to this unit implicational system. The paper also gives a nice characterization of this canonical basis and makes precise its link with the prime implicants of the Horn function associated to a closure operator. It concludes that it is necessary to compare more closely related works made independently, and with a different terminology, in order to take advantage of the really new results in these works.

1. Introduction

In this paper, we deal with "implications", and more detailed explanations are first required for our use of this term. Consider data organized as a set Ω of "objects" (also denoted prototypes, observations, . . .) together with a set S of "attributes" (also denoted characteristics, descriptors, fields, . . .), and where each object is related to a subset of attributes by a binary relation between the objects and the attributes. Such a *data set* appears in several domains, for instance in Data Analysis [18], in Data Mining [25], in Knowledge Spaces [17], in Formal Concept Analysis (FCA, [21]). For example, objects are patients, consumers, students or planets; attributes are symptoms, products, problems, characteristics. Each patient is described by the list of the symptoms he manifests; each consumer is described by the list of products he buys; each student is described by the list of problems he solves; each planet by the list of the characteristics that it possesses. It is convenient to adopt here the FCA's terminology and to call a *context* the triple composed of the set Ω of objects, the set S of attributes and the binary relation R between Ω and S.

When all the consumers buying the two products x and y also buy the product z, or, when all the students solving the two problems x and y also solve the problems z, there is a dependence between x and y on one hand, and z on the other hand. In the general case, there is a dependence between two subsets X and Y of attributes when all objects related to the attributes of X also are related to the attributes of Y. Such a dependence is called a *valid association rule* in Data Mining,

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i.e. an association rule where the proportion of objects related to X and Y among the objects related to X (also called the *confidence*) is equal to 100%. In Formal Concept Analysis, one says that X *implies* Y. It is in this sense that the term *implication* is used in this paper, and an implication between X and Y will be denoted $X \rightarrow Y$. It is clear that these implications between attributes are "contextual" since they depend on the given context.

The theory of relational databases induces the same notion of implication between attributes. Data is organized as relations (according to Codd's terminology in [12]) between a list of "records" and a set of multi-valued attributes. A record is then a tuple of values, one for the domain of values of each attribute. Consider the case where all the records related to the same values on a set X of attributes are also related to the same values on another set Y of attributes. Then in the theory of relational databases one says that Y *functionally depends* on X or that X *determines* Y or that there is a *functional dependency* (FD) between X and Y. It is easy to define a binary relation between the set of all pairs of records and the set of attributes so that Y functionally depends on X if and only if X implies Y with respect to this context (see [21]).

Consider a context and the set of all associated implications between subsets of the set S of attributes. Formally, the implication $X \rightarrow Y$ is an ordered pair (X, Y) of subsets of S. So, the set of all implications between attributes is a binary relation on the power set $\mathcal{P}(S)$ of the attributes. It is useful to consider any binary relation on $\mathcal{P}(S)$ (it will be clear why below). Such an (arbitrary) binary relation on $\mathcal{P}(S)$ is called here an *implicational system* (it is called a *set of implications* in FCA and a *set of functional dependencies* in the relational data model).

It is also useful to consider a *unit implicational system* defined as a binary relation between $\mathcal{P}(S)$ and S. It is clear that one can associate a unit implicational system with an implicational system: any implication $X \rightarrow Y$ can be replaced by the set of unit implications $\{X \rightarrow y\ ,\ y \in Y\}$. Conversely, one can associate an implicational system with a unit implicational system: for instance, the set of implications $X \rightarrow Y = \{y \in S\ :\ X \rightarrow y\}$. Observe that this correspondence is not "one-to-one" (see [21]).

Let us now return to the implicational system associated with a context (Ω, S, R). It is not an arbitrary relation on $\mathcal{P}(S)$. For instance, if $X \rightarrow Y$ and $Y \rightarrow Z$, one also has $X \rightarrow Z$ (check what it means in the context associated with a data set context as well as in the context associated with databases). Such an implicational system is called here a *full implicational system*; in the theory of knowledge structures it is called an *entail relation*, in FCA a *closed set of implications* and in the theory of the relational databases a *full family of functional dependencies* or a *relational databases scheme* or even a *relation scheme* (at least by some authors since the terminology of databases is far from being unified). A fundamental fact first observed by Armstrong in [3] in the theory of relational databases is the following: "there is a one to one correspondence between the set of all the full implicational systems defined on a set S and the set of all closure operators defined on S." These sets are also in a one to one correspondence with many other sets (see [11]) and in particular with the set of all *full unit implicational systems* (called *entailments* in the theory of knowledge structures), the set of all *closure systems* and the set of all *pure Horn (Boolean) functions* (precise definitions and references are given in Section 2).

Now the same problem has been encountered in all the aforementioned domains. Take, for instance, the full family of functional dependencies associated with a table in a relational database. It contains many dependencies but some of them are trivial (for instance, $X \rightarrow Y$ if $Y \subseteq X$) and some can be deduced from others (for instance, if $X \rightarrow y$ and $y \rightarrow z$ one has also $X \rightarrow z$). So one searches for "small" generating implicational systems allowing us to recover a given full implicational system (the definition of a generating system is given in Section 2.2). Observe that thanks to the correspondence between full implicational systems and closure operators, a generating system allows us just as well to recover a closure operator. In this paper we will rather consider that one wants to efficiently recover a closure operator (which can be the closure operator corresponding to a full implicational system).

There exists a significant result on the minimal generation of a closure operator (or of a full implicational system) by an implicational system. It has been obtained independently (and with different formulations) by Maier [32] and Guigues and Duquenne [23]. The generating implicational system obtained is often called the Duquenne–Guigues *canonical basis*. Here we will not be concerned with this basis since our results bear on the generation of a closure operator by a unit implicational system. We will show that five generating unit implicational systems obtained by different authors in different fields and with different formalisms are in fact identical. This unique generating system has properties that justify calling it the *canonical direct unit implicational basis* (but it is not the unit implicational system associated with the Duquenne–Guigues canonical basis). Moreover, finding it is the same as finding the set of the *prime implicants* of a Boolean function.

We end this introduction by presenting the contents of the different sections of the paper. Section 2 recalls the notions about lattices, closure operators or closure systems, and the (unit) implicational systems we will use. In Section 3 we describe the five unit implicational systems proposed by different authors in order to efficiently generate a closure operator (for reasons explained later they are called "bases" of the closure operator). Section 4 contains our main results. We prove that these five bases are the same and thus they define an unique basis which can be called the canonical direct unit implicational basis. Whereas some of these equalities are easy to obtain, others are deduced from a non obvious characterization of a *direct basis*. One of the corollaries of these results shows that the *necessary sets for x* (defined in the context of relational databases) can be identified with the *x-dominating sets* (defined in the context of choice functions in microeconomics). It is (more or less) well known that closure systems on a set S are in a one-to-one correspondence with the so-called *pure Horn Boolean functions* defined on $\mathcal{P}(S)$. In Section 5 we show that finding the canonical direct unit implicational basis is the same as finding the prime implicants (or the prime implicates) of a (pure) Horn Boolean function. Section 6 is for the readers interested by the history of the appearance of some notions considered in this paper and the works relating these notions to traditional notions in logic.

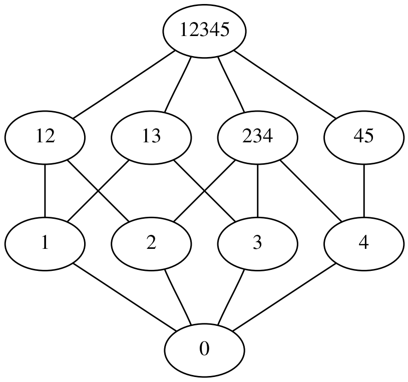

Fig. 1. The lattice $(\mathbb{F}, \subseteq)$ represented by its Hasse diagram, where $\mathbb{F}$ is the closure system of our example.

2. Recalls and definitions

All the sets considered in this paper are finite.

2.1. Set systems and Lattices

A *lattice* is a partially ordered set $(L, \leq)$ such that any pair $\{x, y\}$ of elements of L has a *join* (i.e. a least upper bound) denoted by $x \vee y$ and a *meet* (i.e. a greatest lower bound) denoted by $x \wedge y$. For any classical notion on partially ordered sets or lattices, see, for instance, Caspard, Leclerc and Monjardet [10] or Davey and Priestley [14].

A *set system* on a set S is a family of subsets of S. A *closure system* $\mathbb{F}$ on a set S, also called a *Moore family*, is a set system stable by intersection and which contains S: $S \in \mathbb{F}$ and $F_1, F_2 \in \mathbb{F}$ imply $F_1 \cap F_2 \in \mathbb{F}$. The subsets belonging to a closure system $\mathbb{F}$ are called the *closed sets* of $\mathbb{F}$. The partially ordered set $(\mathbb{F}, \subseteq)$ is a lattice with, for each $F_1, F_2 \in \mathbb{F}$, $F_1 \wedge F_2 = F_1 \cap F_2$ and $F_1 \wedge F_2 = \bigcap\{F \in \mathbb{F} \mid F_1 \cup F_2 \subseteq F\}$. It's well known that any lattice L is isomorphic to the lattice of closed sets of a closure system [8].

Example 1. Consider the closure system[1] on the set $S = \{1, 2, 3, 4, 5\}$:

$$\mathbb{F} = \{\emptyset, 1, 2, 3, 4, 12, 13, 45, 234, S\}$$

One can verify that it is stable by intersection. The lattice $(\mathbb{F}, \subseteq)$ is represented by its Hasse diagram in Fig. 1. We will use this example to illustrate several notions in this paper.

A *closure operator* on a set S is a map φ on $\mathcal{P}(S)$ satisfying, $\forall X, Y \subseteq S$:

$$X \subseteq \varphi(Y) \Leftrightarrow \varphi(X) \subseteq \varphi(Y) \tag{1}$$

Equivalently, and more commonly, a closure operator is defined as a map φ satisfying the three following properties: φ is *isotone* (i.e. $\forall X, X' \subseteq S, X \subseteq X' \Rightarrow \varphi(X) \subseteq \varphi(X')$), *extensive* (i.e. $\forall X \subseteq S, X \subseteq \varphi(X)$) and *idempotent* (i.e. $\forall X \subseteq S$, $\varphi^2(X) = \varphi(X)$). Still equivalently, a closure operator is an extensive map satisfying the *path-independence* property (i.e. $\forall X, Y \subseteq S, \varphi(X \cup Y) = \varphi(\varphi(X) \cup Y)$). The set $\varphi(X)$ is called the *closure* of X by φ. The set X is said to be *closed* by φ whenever it is a fixed point of φ, i.e. when $\varphi(X) = X$.

Closure operators are in one-to-one correspondence with closure systems. On the first hand, the set of all closed elements of φ forms a closure system $\mathbb{F}_\varphi$:

$$\mathbb{F}_\varphi = \{F \subseteq S \mid F = \varphi(F)\} \tag{2}$$

Dually, given a closure system $\mathbb{F}$ on a set S, one defines the closure $\varphi_{\mathbb{F}}(X)$ of a subset X of S as the least element $F \in \mathbb{F}$ that contains X:

$$\varphi_{\mathbb{F}}(X) = \bigcap\{F \in \mathbb{F} \mid X \subseteq F\} \tag{3}$$

Moreover for all $F_1, F_2 \in \mathbb{F}$, $F_1 \vee F_2 = \varphi_{\mathbb{F}}(F_1 \cup F_2)$ and $F_1 \wedge F_2 = \varphi_{\mathbb{F}}(F_1 \cap F_2) = F_1 \cap F_2$.

A subset B of S is a *basis* of F, with F closed set for φ, if $\varphi(B) = F$ and $\varphi(A) \subset \varphi(B)$ for every $A \subset B$ (in other words, B is a *minimal generating set* of F). A subset B of S is *free* if for every $x \in B$ $x \notin \varphi(B \backslash x)$. Or, equivalently, B is free if and only if $\varphi(A) \subset \varphi(B)$ for every $A \subset B$, or if and only if B is a basis of $\varphi(B)$. An element x of a subset X of S is an *extreme point* of X if $x \notin \varphi(X \setminus x)$. One denotes by $\mathbb{E}x_\varphi(X)$ or simply $\mathbb{E}x(X)$ the set of extreme points of X. Observe that X is free if and only if $\mathbb{E}x(X) = X$. A subset C of S is a *copoint* of $x \in S$ if C is a maximal subset of S such that $x \notin \varphi(C)$. It is well known that in the lattice $\mathbb{F}_\varphi$, the copoints of x are *meet-irreducible* closed sets (i.e. cannot be obtained as meet of closed sets different from themselves).

[1] In this example as in the following, a subset $X = \{x_1, x_2, \ldots, x_n\}$ is written as the word $x_1x_2 \ldots x_n$. Moreover, we abuse notation in the following and use $X + x$ (respectively, $X \setminus x$) for $X \cup \{x\}$ (respectively, $X \backslash \{x\}$), with $X \subseteq S$ and $x \in S$.

2.2. Unit Implicational System

A *Unit Implicational System* (UIS for short) Σ on S is a binary relation between $\mathcal{P}(S)$ and S: $\Sigma \subseteq \mathcal{P}(S) \times S$. An ordered pair $(A, b) \in \Sigma$ is called a Σ*-implication* whose *premise* is A and *conclusion* is b. It is written $A \to_\Sigma b$ or $A \to b$ (meaning "A implies b"). A subset $X \subseteq S$ *respects* a Σ-implication $A \to b$ when $A \subseteq X$ implies $b \in X$ (i.e. "if X contains A then X contains b").

$X \subseteq S$ is Σ*-closed* when X respects all Σ-implications, i.e $A \subseteq X$ implies $b \in X$ for every Σ-implication $A \to b$. The set of all Σ-closed sets forms a closure system $\mathbb{F}_\Sigma$ on S:

$$\mathbb{F}_\Sigma = \{X \subseteq S \mid X \text{ is } \Sigma\text{-closed}\} \tag{4}$$

Then, we can associate with Σ a closure operator $\varphi_\Sigma = \varphi_{\mathbb{F}_\Sigma}$. One can state [43] that φ_Σ is the closure operator obtained by the iteration of the following isotone and extensive map, with $X \subseteq S$:

$$\varphi_\Sigma(X) = \pi_\Sigma(X) \cup \pi^2_\Sigma(X) \cup \pi^3_\Sigma(X) \cup \dots \tag{5}$$

where

$$\pi_\Sigma(X) = X \cup \bigcup \{b \mid A \subseteq X \text{ and } A \to_\Sigma b\} \tag{6}$$

and

$$\pi^2_\Sigma(X) = \pi_\Sigma(X) \cup \bigcup \{b \mid A \subseteq \pi_\Sigma(X) \text{ and } A \to_\Sigma b\} \tag{7}$$

Observe that the procedure in (5) terminates since S is finite. Moreover, $\varphi_\Sigma(X) = \pi^n_\Sigma(X)$ with $n \leq |S|$ being the first integer such that $\pi^n_\Sigma(X) = \pi^{n+1}_\Sigma(X)$, and it is well known that iteration of an isotone and extensive map defined on a finite set leads to an idempotent map, i.e. a closure operator.

Now, consider a closure operator φ on S. Then the closed sets of φ coincide with the Σ-closed sets of the following UIS:

$$\Sigma_\varphi = \{X \to y \mid y \in \varphi(X) \text{ and } X \subseteq S\} \tag{8}$$

It is easy to see that Σ_φ satisfies the two following properties:

F1 $x \in X \subseteq S$ implies $X \to_{\Sigma_\varphi} x$.
F2 for every $y \in S$ and all $X, Y \subseteq S$, $[X \to_{\Sigma_\varphi} y$ and $\forall x \in X, Y \to_{\Sigma_\varphi} x]$ implies $Y \to_{\Sigma_\varphi} y$.

Unit IS satisfying properties F1 and F2 are called full UISs and are in one-to-one correspondence with closure operators, and thus with closure systems and lattices.

The set of all full UISs is itself a closure system defined on the set of UISs. So, when a UIS Σ is not full, there exists a least full UIS containing it. This full UIS is nothing other than Σ_φ where $\varphi = \varphi_\Sigma$ is the closure operator associated with Σ (see Equation 5). This full UIS Σ_φ can be obtained by applying recursively rules F1 and F2 to Σ. The UIS Σ is then called a *generating system* (or *cover* in relational data bases) for the full UIS Σ_φ, and thus for the induced closure operator φ, the closure system $\mathbb{F}_\Sigma$, and the induced lattice $(\mathbb{F}_\Sigma, \subseteq)$. When some UISs Σ and Σ' on S are generating systems for the same closure system, they are called *equivalent* (i.e. $\mathbb{F}_\Sigma = \mathbb{F}_{\Sigma'}$).

An illustration of a generating system of a full UIS Σ_φ is given by the UIS Σ_{free} composed of the subsets of S that also are free subsets:

$$\Sigma_{free} = \{X \to y \ : \ y \in \varphi(X) \backslash X \text{ and } X \text{ free subset of } S\} \tag{9}$$

An UIS Σ is called *direct* or *iteration-free* if for every $X \subseteq S$, $\varphi_\Sigma(X) = \pi_\Sigma(X)$ (see Eq. (6)). An UIS Σ is *minimal* or *non-redundant* if $\Sigma \setminus \{X \to y\}$ is not equivalent to Σ, for all $X \to y$ in Σ. It is *minimum* if it is of least cardinality, i.e. if $|\Sigma| \leq |\Sigma'|$ for all UISs Σ' equivalent to Σ. A minimum UIS is trivially non-redundant, but the converse is false. Σ is *optimal* if $s(\Sigma) \leq s(\Sigma')$ for all UISs Σ' equivalent to Σ, where the *size* $s(\Sigma)$ of Σ is defined by:

$$s(\Sigma) = \sum_{A \to b \in \Sigma} (|A| + 1) \tag{10}$$

A minimal UIS is usually called a *basis* for the induced closure system (and thus for the induced lattice), and a *minimum basis* is then a basis of least cardinality. An implication $X \to_\Sigma x$ with $x \in X$ is called *trivial*. An UIS is called *proper* if it does not contain trivial implications. When an UIS is not proper, an equivalent proper UIS can be obtained by applying the following rule:

F3 delete $A \to_\Sigma b$ from Σ when $b \in A$.

Example 2. Consider the closure system of our example given by the lattice $(\mathbb{F}, \subseteq)$ in Fig. 1 and the generating system Σ_{free}:

$$\Sigma_{free} = \begin{cases} (\ 1\ )\ 5 \to 4 & (\ 2\ )\ 23 \to 4 & (\ 3\ )\ 24 \to 3 & (\ 4\ )\ 34 \to 2 \\ (\ 5\ )\ 14 \to 2 & (\ 6\ )\ 14 \to 3 & (\ 7\ )\ 14 \to 5 & (\ 8\ )\ 25 \to 1 \\ (\ 9\ )\ 35 \to 1 & (10)\ 15 \to 2 & (11)\ 35 \to 2 & (12)\ 15 \to 3 \\ (13)\ 25 \to 3 & (14)\ 123 \to 5 & (15)\ 15 \to 4 & (16)\ 25 \to 4 \\ (17)\ 35 \to 4 & (18)\ 123 \to 4 & & \end{cases}$$

Notice that Σ_{free} is a proper UIS since for every implication, the conclusion is not included in the premise. Concerning the direct property, it is clear that Σ_{free} is a direct UIS.

Remark. In the following sections, we will assume that all the no proper UISs have been replaced by an equivalent proper UIS (by applying the above rule F3). Then (except for Proposition 4 below), the term UIS will always mean proper UIS.

3. Some interesting bases

In this section we are going to define several UISs which are generating systems for a given closure operator φ (equivalently for a given closure system $\mathbb{F}$) which can be the closure operator associated with a given UIS Σ. In the literature on IS, the term basis is often used not only for minimal IS but also for IS satisfying various minimality criteria. We will do the same by defining five such bases.

3.1. The direct-optimal basis Σ_{do}

A number of problems related to closure systems, (thus closure operators, lattices or implicational systems) can be answered by computing closures of the type $\varphi_\Sigma(X)$, for some $X \subseteq S$. According to the definition (see Eq. (5)) $\varphi(X)$ can be obtained given an UIS Σ by iteratively scanning Σ-implications: $\varphi(X)$ is initialized with X then increased with b for each implication $A \to_\Sigma b$ such that $\varphi(X)$ contains A. The computation cost depends on the number of iterations and in any case is bounded by $|S|$. It is worth noticing that for direct (or iteration-free) UISs the computation of $\varphi(X)$ requires only one iteration, since $\varphi_\Sigma(X) = \pi_\Sigma(X)$. The *direct-optimal* property combines the directness and optimality properties:

Definition 3. A UIS Σ is *direct-optimal* if it is direct, and if $s(\Sigma) \le s(\Sigma')$ for any direct UIS Σ' equivalent to Σ.

In [6], Bertet and Nebut show that a direct-optimal UIS is unique and can be obtained from any equivalent UIS:

Proposition 4. *[6] The* direct-optimal basis Σ_{do} *is obtained from any equivalent UIS Σ as follows:*

(1) *first apply recursively the following rule[2] to obtain a direct equivalent UIS:*
F7 *for all $A \to_\Sigma b$ and $C + b \to_\Sigma d$ with $d \neq b$, add $A \cup C \to d$ to Σ*
(2) *then apply the F3 rule to obtain a proper UIS, and the following rule to minimize premises of the Σ-implications:*
F8 *for all $A \to_\Sigma b$ and $C \to_\Sigma b$, if $C \subset A$ then delete $A \to_\Sigma b$ from Σ.*

Example 5. Consider our example given by $(\mathbb{F}, \subseteq)$ in Fig. 1. The basis Σ_{do} is:

$$\Sigma_{do} = \begin{cases} (1)\,5 \to 4 & (2)\,23 \to 4 & (3)\,24 \to 3 & (4)\,34 \to 2 \\ (5)\,14 \to 2 & (6)\,14 \to 3 & (7)\,14 \to 5 & (8)\,25 \to 1 \\ (9)\,35 \to 1 & (10)\,15 \to 2 & (11)\,35 \to 2 & (12)\,15 \to 3 \\ (13)\,25 \to 3 & (14)\,123 \to 5 & & \end{cases}$$

One can verify that Σ_{do} is direct like Σ_{free}. Moreover, $s(\Sigma_{do}) < s(\Sigma_{free})$ and $\Sigma_{do} \subset \Sigma_{free}$.

3.2. The dependence relation's basis Σ_{dep}

The dependence relation's basis Σ_δ on S comes from the *dependence relation* δ defined for a lattice, and introduced in [35].

Definition 6. The *dependence relation's basis* Σ_δ is:

$$\Sigma_\delta = \{X + y \to x \,:\, x\delta_X y \text{ and } X \text{ is minimal for this property}\} \tag{11}$$

where the *dependence relation* δ_X is defined on S, with $x, y \in S$ and $X \subset S$, by:

$$x\delta_X y \text{ if and only if } x \notin \varphi(X),\ y \notin \varphi(X) \text{ and } x \in \varphi(X + y) \tag{12}$$

The dual relation of the relation δ_X has been considered in [4] where it is called *domination*. One can observe that the *dependence relation* δ on the lattice $(\mathbb{F}, \subseteq)$ is then given by $x\delta y$ if there exists $X \subseteq S\backslash\{x, y\}$ such that $x\delta_X y$ (so $\delta = \cup\{\delta_X,\ X \subset S\}$).

Example 7. Fig. 2 gives the dependence relations δ and δ_X of our example, where two vertices x and y are linked by an arc if $x\delta y$. This arc is valued by the subsets X such that $x\delta_X y$. For instance, $5\delta_4 1$, and $5\delta_{23} 1$.

3.3. The canonical iteration-free basis Σ_{cif}

The canonical iteration-free basis on S is an implicational system introduced by Wild in [43]. As mentioned in the introduction, this implicational system can be transformed into a unit implicational system denoted Σ_{cif}:

Definition 8. The unit basis Σ_{cif} deduced from the *canonical iteration-free basis* is:

$$\Sigma_{cif} = \{B \to x \,:\, x \in \varphi(B)\backslash\pi_\varphi(B) \text{ and } B \text{ is a free subset}\} \tag{13}$$

where π_φ is derived from φ as follows:

$$\pi_\varphi(B) = B \cup \{x \in S \ :\ \text{there exists } A \subset B \text{ with } x \in \varphi(A)^3\}$$

[2] When Σ is not proper, this rule has to be applied only when $b \notin A$ and $d \notin A \cup C$.
[3] When B is not a free subset, the condition $\varphi(A) \subset \varphi(B)$ has to be added.

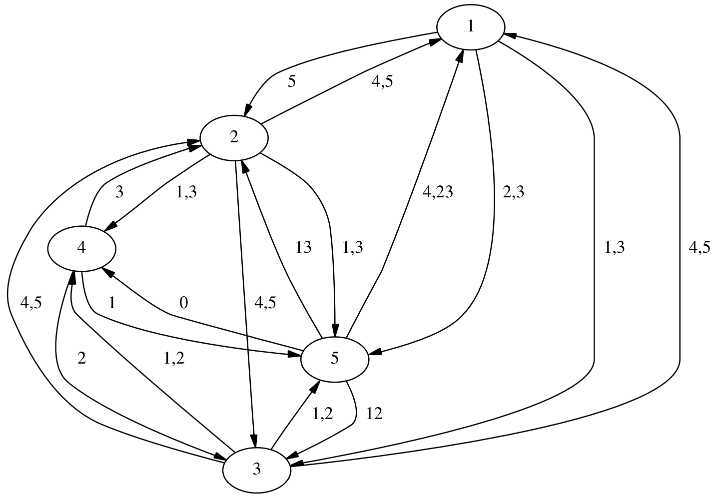

Fig. 2. Relation δ_X for $\mathbb{F}$ of our example represented by a directed graph where each relation $a\delta_X b$ is represented by an arc and labeled by X ($\emptyset$ is denoted by 0).

3.4. The left-minimal basis Σ_{lm}

The *left-minimal basis* Σ_{lm} is the restriction of the full UIS Σ_φ to implications where the premise is of minimal cardinality. Using the definition of Σ_φ (see Definition 8), Σ_{lm} can be expressed directly from φ:

Definition 9. The *left-minimal basis* Σ_{lm} is:

$$\Sigma_{lm} = \{X \to y \;:\; y \in \varphi(X) \setminus X \text{ and for every } X' \subset X,\; y \notin \varphi(X')\} \tag{14}$$

An implication $X \to y$ is called *left-minimal* when it is a Σ_{lm}-implication. It is also called a *proper implication* in [41] where implications are used in the data-mining area research, and *minimal functional dependency* in the domains of relational databases and Horn theories [32,30].

Example 10. For our example, Σ_{lm} is the same as Σ_{do}. Remark that Σ_{lm} of our example has 14 implications, and not 15 as incorrectly written in [11] about the same example (p.37).

3.5. The weak-implication basis Σ_{weak}

The *weak-implication basis* has been introduced by Rusch and Wille in [38] to show a connection between the theory of knowledge spaces [17] and formal concept analysis [21]. It is based on the definition of a copoint (recall that a subset C of S is a copoint of $x \in S$ if C is a maximal subset of S such that $x \notin \varphi(C)$), and on the following classical notion of a transversal set.

A subset B of a set S is a *transversal* of a family $\mathcal{F}$ of subsets of S if $B \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. A transversal B is a *minimal transversal* of $\mathcal{F}$ if for every $A \subset B$, A is not a transversal of $\mathcal{F}$ (i.e. there exists $F \in \mathcal{F}$ with $A \cap F = \emptyset$).

Definition 11. [38] The *weak-implication basis* Σ_{weak} is:

$$\Sigma_{weak} = \{B \to x \;:\; B \subseteq S \text{ and } B \text{ is a blockade for } x\} \tag{15}$$

where a *blockade* for $x \in S$ (also called *x-block*) is a minimal transversal of $\mathcal{D}_x$, the following family of subsets of S:

$$\mathcal{D}_x = \{S \backslash (C + x)\;,\; C \text{ is a copoint of } x\} \tag{16}$$

Lemma 12. *Let $x \in S$ and $B \subseteq S$. Then the B x-block implies $x \notin B$ and $x \in \varphi(B)$ (i.e. $B \to x$).*

Proof. Consider an x-block $B \subseteq S$. The first point is immediate: by definition of a blockade for x, we have $x \notin B$. For the second point, suppose $x \notin \varphi(B)$. Let $F \subseteq S$ be a maximal closed set of φ such that $x \notin F$ and $\varphi(B) \subseteq F$. Then F is a copoint of x. However, $B \subseteq F$ implies $B \cap (S \backslash (F + x)) = \emptyset$, a contradiction with B, an x-block. $\square$

4. The main results

The main result (Theorem 15) of this paper is to state the equality between the five bases defined in the previous section all of which are thus direct bases. The second main result (Theorem 14) is to give an interesting characterization of the direct property based on an *exchange property*. This exchange property has been independently introduced in [16] and in a stronger form in [6]. In [16], Demetrovics and Nam Son use it to define the notion of Sperner village and to show its equivalence with the notion of closure operator. In [6], Bertet and Nebut use it in the generation of the direct-optimal basis Σ_{do} where rule F7 results directly from this exchange property.

The characterization of Theorem 14 uses another formulation of the direct property issued from the definition (i.e. for every $X \subseteq S$, $\varphi(X) = \pi_\Sigma(X)$).

Lemma 13. *An UIS Σ is direct if and only if for every $X \subseteq S$, $\pi_\Sigma(X) = \pi^2_\Sigma(X)$.*

Theorem 14. *An UIS Σ is direct if and only if it satisfies the following* exchange condition*: $\forall A, C \subseteq S$, $\forall b \in S \backslash A$, $\forall d \in S \backslash (A \cup C)$,*

$$A \to_\Sigma b \text{ and } C + b \to_\Sigma d \text{ imply there exists } G \subseteq A \cup C \text{ such that } G \to_\Sigma d \tag{17}$$

Proof. $\Rightarrow$: Let Σ be a direct (or not) UIS. Assume that for $b \in S \backslash A$ and $d \in S \backslash (A \cup C)$, we have $A \to_\Sigma b$ and $C + b \to_\Sigma d$, which means $b \in \varphi_\Sigma(A)$ and $d \in \varphi_\Sigma(C + b)$. Then, using the isotone path-independence properties of a closure operator, we get

$$d \in \varphi_\Sigma(A \cup (C + b)) = \varphi_\Sigma(\varphi_\Sigma(A + b) \cup C) = \varphi_\Sigma(\varphi_\Sigma(A) \cup C) = \varphi_\Sigma(A \cup C)$$

Now, there exists $G \subseteq A \cup C$ such that $G \to_\Sigma d$.
$\Leftarrow$: Let Σ be a UIS satisfying condition (17). One must show that $\varphi_\Sigma(X) = \pi_\Sigma(X)$, or equivalently by Lemma 13 that $\pi_\Sigma(X) = \pi^2_\Sigma(X)$, or still equivalently (since π_Σ is extensive) that $\pi^2_\Sigma(X) \subseteq \pi_\Sigma(X)$.
Assume that there exists X with $\pi_\Sigma(X) \subset \pi^2_\Sigma(X)$, i.e. that there exists $z \in \pi^2_\Sigma(X) \backslash \pi_\Sigma(X)$. Then there exists $Z \subseteq \pi_\Sigma(X)$ with $Z \to_\Sigma z$. We set $p(Z) = |Z \cap (\pi_\Sigma(X) \backslash X)|$. The proof of $\varphi_\Sigma(X) = \pi_\Sigma(X)$ will follow immediately from the proof of the following result:

if $p(Z) = p$ then there exists $Z' \subseteq S$ with $Z' \to_\Sigma z$ and $p(Z') < p(Z)$.

Indeed, by iteration of this result we would get some $Z^{(k)}$ with $Z^{(k)} \to_\Sigma z$ and $p(Z^{(k)}) = 0$, which means $Z^{(k)} \subseteq X$ and $z \in \pi_\Sigma(X)$, a contradiction with our hypothesis.
First, observe that $p(Z) > 0$: if not, $Z \subset X$ and $z \in \pi_\Sigma(X)$, a contradiction. $p(Z) > 0$ means that there exists $y \in Z$ with $y \in \pi_\Sigma(X) \backslash X$. Thus there exists $Y \subseteq X$ with $Y \to_\Sigma y$. Now writing $Z = U + y$, we have $Y \to_\Sigma y$, $U + y \to_\Sigma z$ with $y \notin Y$ and (since $z \notin \pi_\Sigma(X)$) $z \notin Y \cup U$. So, by applying the exchange condition, we get that there exists $Z' \subseteq Y \cup U$ with $Z' \to_\Sigma z$. Moreover, since $p(Y \cup U) = p(Z) - 1$, we have $p(Z') < p(Z)$ as desired. □

Now, let us give our other main result.

Theorem 15. *Let φ be a closure operator defined on a set S, and the five associated UISs above defined. Then*

$$\Sigma_{do} = \Sigma_{cif} = \Sigma_{dep} = \Sigma_{lm} = \Sigma_{weak}$$

Proof. We prove first $\Sigma_{cif} = \Sigma_{dep} = \Sigma_{lm} = \Sigma_{weak}$ by proving $\Sigma_{cif} \subseteq \Sigma_{dep} \subseteq \Sigma_{lm} \subseteq \Sigma_{weak} \subseteq \Sigma_{cif}$. Then we prove $\Sigma_{do} = \Sigma_{lm}$

$\Sigma_{cif} \subseteq \Sigma_{dep}$: Let $B \to x$ be a Σ_{cif}-implication. This means that $x \in \varphi(B) \backslash \pi_\varphi(B)$ where B is free, i.e. $x \in \varphi(B)$ and $x \notin \varphi(A)$ for every $A \subset B$. Take any y in B. Since $B \backslash y \subset B$ and B is free, one has $x \notin \varphi(B \backslash y)$, $y \notin \varphi(B \backslash y)$ and (obviously) $x \in \varphi((B \backslash y) + y)$. If $X \subset B \backslash y$, $X + y \subset B$, and so $x \notin \varphi(X + y)$. Then $B \backslash y$ is minimal such that $x, y \notin \varphi(X)$ and $x \in \varphi(X + y)$, i.e. $B \to x$ is a Σ_{dep}-implication.

$\Sigma_{dep} \subseteq \Sigma_{lm}$: Let $B = X + y \to x$ be a Σ_{dep}-implication. Then $x \notin \varphi(X)$ and for every $Y \subset X$, $x \notin \varphi(Y + y)$. So $B \to x$ is a Σ_{lm}-implication.

$\Sigma_{lm} \subseteq \Sigma_{weak}$: Let $B \to x$ be a Σ_{lm}-implication. Let us first prove that B is a transversal of $\mathcal{D}_x = \{S \backslash (C + x), C \text{ copoint of } x\}$ before to prove that it is a minimal transversal. Since $x \notin B$, B is a transversal of $\mathcal{D}_x$ if and only if B is a transversal of $\mathcal{D}'_x = \{S \backslash C, C \text{ copoint of } x\}$. Suppose there exists C, a copoint of x such that $B \cap (S \backslash C) = \emptyset$ and so $B \subseteq C$. Then $\varphi(B) \subseteq C$ which implies $x \in C$, a contradiction with C, a copoint of x.
Suppose now that B is not a minimal transversal of $\mathcal{D}_x$., i.e. that there exists $Y \subset B$ with the Y transversal of $\mathcal{D}_x$. Since B is left-minimal for the implication $B \to x$, we have $x \notin \varphi(Y)$. Then there exists a copoint C of x such that $Y \subseteq \varphi(Y) \subseteq C$. Therefore $Y \cap (S \backslash C) = \emptyset$, a contradiction with Y transversal of $\mathcal{D}_x$.

$\Sigma_{weak} \subseteq \Sigma_{cif}$: Let $B \to x$ be a Σ_{weak}-implication. This means that $x \in \varphi(B) \backslash B$ and B is minimal transversal of $\mathcal{D}_x = \{S \backslash (C + x), C \text{ copoint of } x\}$. We prove first that B is free by showing that for any $A \subset B$ one has $\varphi(A) \subset \varphi(B)$. Indeed, when $A \subset B$, A is not a transversal of $\mathcal{D}_x$ and there exists a copoint C of x such that $A \cap (S \backslash C + x) = \emptyset$. So $A \subseteq C$ (since $x \notin A$) and $\varphi(A) \subseteq C$. However, $x \notin C$ implies $x \notin \varphi(A)$ and so $\varphi(A) \subset \varphi(B)$. Moreover, we have just proved that $x \notin \varphi(A)$ for every $A \subset B$, i.e. that $x \notin \pi_\varphi(B)$. Finally, $B \to x$ is a Σ_{cif}-implication.

$\Sigma_{lm} = \Sigma_{do}$: To prove the equality $\Sigma_{lm} = \Sigma_{do}$, let us prove that Σ_{lm} is direct-optimal (since there is a unique direct-optimal basis). First we prove that Σ_{lm} is direct, i.e. that for every $A \subseteq S$, $\varphi(A) = A \cup \{x \in S : \text{there exists } B \subseteq A \text{ with } B \to_{\Sigma_{lm}} x\}$. This is obvious since one can take for B a basis of $\varphi(A)$ such that $B \subseteq A$.
Now, let us prove that Σ_{lm} is direct-optimal. Consider a direct and equivalent UIS Σ. It is sufficient to prove that, when $B \to x$ is a Σ_{lm}-implication, it is also a Σ-implication. Assume that it is not the case. Since $B \to x$ is left-minimal, $A \to x \notin \Sigma$ for every $A \subset B$. Therefore, $x \notin \varphi(B) = B \cup \{x \in S : \text{there exists } A \subseteq B \text{ with } A \to_\Sigma x\}$, a contradiction with Σ direct. □

The above result justifies the following definition:

Definition 16. The unique basis obtained in Theorem 15 is called the *canonical direct unit basis*, and is denoted by Σ_{cd}.

Theorems 14 and 15 induce other nice characterizations of the canonical direct unit basis:

Corollary 17. *Let φ be a closure operator. The canonical direct unit basis Σ_{cd} is the smallest basis of the set of all direct unit bases ordered by inclusion.*

Indeed, $\Sigma_{cd} = \Sigma_{cif}$ and the result comes immediately from the property of canonicity proved by Wild [43] for the non unit direct basis associated with the free subsets. Indeed, this property says that if $X \to Y$ is any implication of this canonical basis, then any other direct basis contains implications $X \to Y_i$ such that $Y \subseteq \bigcup Y_i$.

Corollary 18. *An UIS Σ is the canonical direct unit basis if and only if it satisfies the two following properties:*

(1) *for every $x \in S$, $B \to_\Sigma x$ and $B' \to_\Sigma x$, B and B' are incomparable.*
(2) *the exchange condition (Eq. (17)).*

Indeed, $\Sigma_{cd} = \Sigma_{lm}$. One can observe that the first property in Corollary 18 can equivalently be reformulated using the terminology of a Sperner family like in [16]: for every $x \in S$, the set $\mathcal{B}_x$ of all premises of the Σ-implications $B \to_\Sigma x$ forms a Sperner family. The fact that $\Sigma_{lm} = \Sigma_{weak}$ shows that the Sperner family $\mathcal{B}_x$ is the family of blockades of x, i.e. the family of minimal transversals of the family $\mathcal{D}_x = \{S \setminus (C + x) \ : \ C \text{ copoint of } x\}$. We show now that the *necessary sets for* x, and the *x-dominating sets* introduced in the literature are the same that the sets $S \setminus (C + x)$. Mannila and Raiha [33] define a *necessary set* for x as a minimal transversal of $\mathcal{B}_x$. On the other hand, one finds in Aizerman and Aleskerov's book on choice functions [1], the definition of an *x-dominating set* as a subset T of S such that $x \in \mathbb{E}x_\varphi(S \setminus T)$ and $x \notin \mathbb{E}x_\varphi(U)$ for every U satisfying $S \setminus T \subset U$ (recall that $x \in \mathbb{E}x_\varphi(X)$ if $x \notin \varphi(X \backslash x)$).

Corollary 19. *Let φ be a closure operator on S, $T \subseteq S$ and $x \in S \setminus T$. The three following conditions are equivalent:*

(1) *T is a necessary set for x,*
(2) *there exists a copoint C of x such that $T = S \setminus (C + x)$,*
(3) *T is an x-dominating set.*

Proof. $1 \Leftrightarrow 2$ Let us denote by $\mathcal{M}_x$ the family of necessary sets for x. By definition, $\mathcal{M}_x = Tr(\mathcal{B}_x)$, the family of minimal transversals of $\mathcal{B}_x$. As said above, $\mathcal{B}_x = Tr(\mathcal{D}_x)$ the family of minimal transversals of $\mathcal{D}_x = \{S \setminus (C + x) \ : \ C \text{ copoint of } x\}$. However, it is well known that, when $\mathcal{F}$ is a Sperner family, $Tr(Tr(\mathcal{F})) = \mathcal{F}$. Therefore $\mathcal{M}_x = Tr(\mathcal{B}_x) = Tr(Tr(\mathcal{D}_x)) = \mathcal{D}_x$.

$2 \Rightarrow 3$ If $T = S \setminus (C + x)$, one has $S \setminus T = C + x$. Since C is a maximal set such that $x \notin C$, $x \in \mathbb{E}x(S \setminus T)$, whereas if $U \supset S \setminus T = C + x$, then $U \setminus x \supset C$ and $x \notin \mathbb{E}x(U)$.

$3 \Rightarrow 2$ Let T be an x-dominating set. So, $x \in \mathbb{E}x_\varphi(S \setminus T)$, i.e. $\{x \in \varphi((S \setminus T) \setminus x))$. Now, if $U \in S \setminus T$, $U \setminus x \in (S \setminus T) \setminus x$ and $x \in \mathbb{E}x_\varphi(U)$ means that $x \in \varphi(U \setminus x))$. Thus $(S \setminus T) \setminus x) = (S \setminus T + x)$ is a maximal set such that $x \in \varphi(S \setminus T + x))$, i.e. a copoint C of x. Then $T = S \setminus (C + x)$, with C, a copoint of x. □

5. The canonical direct unit basis and the Horn functions

It is well known that the families of subsets of a set S are in a one-to-one correspondence with the Boolean functions defined on the Boolean algebra $\mathcal{P}(S)$. Indeed, one can associate with a family $\mathcal{F}$ of subsets of S its characteristic function $f_\mathcal{F}$:

$$f_\mathcal{F}(M) = \begin{cases} 1 & \text{if } M \in \mathcal{F} \text{ with } M \subseteq S \\ 0 & \text{if not} \end{cases} \tag{18}$$

Conversely, one can associate with a Boolean function f from $\mathcal{P}(S)$ to $\{0, 1\}$ the following family of subsets of S called the *models* or the *true points* of f:

$$\mathcal{F}_f = \{M \subseteq S \ : \ f(M) = 1\} \tag{19}$$

By considering dually the *false points*, one can provide another one-to-one correspondence between families on S and Boolean functions on $\mathcal{P}(S)$. In the following, we will prefer this second correspondence that associates with a Boolean function h the family $\mathcal{F}_h$ of its *false points* or its *counter-models*:

$$\mathcal{F}_h = \{M \subseteq S \ : \ h(M) = 0\} \tag{20}$$

Conversely, one can associate with a family $\mathcal{F}$ on S the Boolean function $h_\mathcal{F}$:

$$h_\mathcal{F}(M) = \begin{cases} 0 & \text{if } M \in \mathcal{F} \text{ with } M \subseteq S \\ 1 & \text{if not} \end{cases} \tag{21}$$

A less known and still less used fact is that the closure systems on S are in a one-to-one correspondence with the Boolean functions called *pure (or definite) Horn functions* (see historical notes for references). Then, any result on closure systems (or closure operators or implicational systems) can be translated into results about Horn functions, and conversely. In this section we are going to do this translation for the canonical direct unit basis.

Unfortunately the terminology used for Boolean functions is not unified. Here we use those employed in [13]. If necessary, the reader will also find in this reference the definitions of all the classical notions for these functions, namely literal, term, clause, disjunctive normal form (DNF), conjunctive normal form (CNF), prime implicant (or implicate).

First we recall now what is called a pure (or definite) Horn function. A term is called *Horn* if it contains exactly one complemented literal. For instance, $34'5$ is a *Horn term* defined on the set of variables $\{1, 2, \ldots, n\}$. A DNF is called Horn is all its terms are Horn. A Boolean function is called a *Horn function* if it can be represented by a Horn DNF. Now we have the following well known result (see Section 6):

Theorem 20. *A Boolean function h of n variables $x_1, x_2, \ldots x_n$ is a Horn function if and only if the set of its false points is a closure system on $S = \{x_1, x_2, \ldots x_n\}$.*

Remark. In the literature one also finds another definition of a Horn function. A clause is called Horn if it contains exactly one non-complemented literal. For instance, $1 \vee 2' \vee 4' \vee 5'$ is a *Horn clause*. A CNF is called Horn if all its clauses are Horn. A Boolean function is called a *Horn function* if it can be represented by a Horn CNF. This definition is not equivalent to the previous one. In fact, a *Boolean function f* is a Horn function in this second sense if and only if the complementary function f' (in the Boolean algebra of all Boolean functions) is Horn in the first sense. With this second definition, one has: *"a Boolean function is a Horn function if and only if the set of its true points is a closure system"*.

Now we can state the relationship between the prime implicants of a Horn function h and the canonical direct unit implicational basis Σ_{cd} of its associated closure operator. It is known that the prime implicants of a Horn function are Horn terms, and so we can write Bx' for such a prime implicant, where B is the subset of S corresponding to the non-complemented literals of this prime implicant. For completeness we give the proof of the following known result (see, for instance, Theorem 4.1 in [30] where the result is proved with the Σ_{lm} version of the canonical direct unit basis).

Proposition 21. *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of elements, and:*

- *h be a Horn function of n variables on $\mathcal{P}(S)$;*
- *$\mathcal{F}_h$ the closure system defined on S by the false points of h;*
- *φ_h the associated closure operator on S;*
- *Σ_{cd} the corresponding canonical direct unit implicational basis.*

Then Bx' is a prime implicant of h if and only if $B \to x \in \Sigma_{cd}$.

Proof. Let Bx' be a prime implicant of h and consider the implication $B \to x$. It belongs to Σ_φ since $h(\varphi_h(B)) = 0$ implies $Bx'(\varphi_h(B)) = 0$ and so $x \in \varphi_h(B)$. Let $A \subset B$. Since Ax' is not an implicant of h, there exists $X \subseteq S$ such that $Ax'(X) = 1$ and $h(X) = 0$. Then, $A \subseteq X \subseteq S \setminus x$ and $X \in \mathcal{F}_h$ which means that $x \notin \varphi_h(A)$. So, $A \to x \notin \Sigma_\varphi$ and $B \to x \in \Sigma_{cd}$.

Conversely, let $B \to x \in \Sigma_{cd}$ and consider the Boolean term Bx'. For $X \subseteq S$, we have $Bx'(X) = 1$ if and only if $B \subseteq X$ and $x \notin X$. Then $X \in \mathcal{F}_h$ and $h(X) = 1$, which shows $Bx' \leq h$. Moreover, $B \leq h$ since $B(\varphi_h(B)) = 1$ and $h(\varphi_h(B)) = 0$. Similarly, if $A \subset B$, $Ax' \not\leq h$, since $Ax'(\varphi_h(A)) = 1$ and $h(\varphi_h(B)) = 0$. Then Bx' is a prime implicant of h. $\square$

Corollary 22. *There is a one-to-one map between the set of prime implicants of a Horn function and the set of implications in the canonical direct unit basis of the closure operator corresponding to the Horn function.*

Remark. When one considers the definition of a Horn function mentioned in the remark following Theorem 20, one gets a one-to-one map between the set of prime implicates of the Horn function and the set of implications in the canonical direct unit basis of the corresponding closure operator.

Example 23. In our example (Example 1), consider the canonical direct UIS Σ_{cd} (equal to Σ_{do} given in Example 5) associated to the closure system $\mathbb{F}$ defined on $S = \{1, 2, 3, 4, 5\}$. By Proposition 21, $\mathbb{F}$ is the closure system given by the false points of the following Horn function whose prime implicants are deduced from Σ_{cd}:

$$\begin{aligned} h &= 54' \vee 234' \vee 243' \vee 342' \vee 142' \vee 143' \vee 145' \vee \\ &\quad 251' \vee 351' \vee 152' \vee 352' \vee 153' \vee 253' \vee 1235' \end{aligned}$$

For instance, one can verify that $12 \in \mathbb{F}$ is equivalent to $h(12) = 0$; and $14 \notin \mathbb{F}$ is equivalent to $h(14) = 1$.

In all many domains where the notions of IS, closure systems or Horn functions are used, significant problems are to implement efficient algorithms to go from the one of these objects to the corresponding others. For instance, to get the implication bases from IS and to get the closure operator or/and the family of closed sets corresponding to a given IS or Horn function (observe that the family of closed sets can have an exponential size). There is plentiful literature on these subjects. In the case of UIS and of the canonical direct unit implicational basis, let us just mention the works in [5,6,30] or [41].

6. Historical note

We try to give the origins of some notions and results used in this paper. It is well known that the notion of a binary relation on a set arose from works of De Morgan and Peirce in the second half of the 19th century. However, it seems to be more difficult to know who introduced for the first time the notion of binary relation between subsets and elements of a set or used for the first time the notion of a binary relation on the power set of a set. It is clear that such relations can be used in many different contexts. For instance, a binary relation between subsets and elements of a set appears in Hertz's 1927 paper [27] where it formalizes a *consequence* relation, and a relation between elements and subsets of a set appears in Appert's paper [2], where a "contiguity" relation allowing to define a topological space is formalized.

Birkhoff [7] dates back the origin of the notions of closure systems and closure operators to Moore's 1909 paper [37]. Indeed, in this paper Moore, speaking in terms of a property of a class of functions, writes: "let a property satisfied by the class (of all functions) and by the greatest common subclass of subclasses satisfying it. Then this property is extensionally attainable in the sense that for every subclass S there exists a least extensive class containing *S*, given by the intersection of all subclasses containing *S*." However, it is probable that Moore's observations about the equivalence of these two notions would have been forgotten if these two concepts, under various names and in a more or less general way, had not played a significant role in the birth of the general topology as an axiomatic theory, in the beginning of the last century. Many mathematicians (Alexander, Alexandroff, Frechet, Hausdorff, Kuratowski, Riesz, Sierpinski, Siskorski, Monteiro, Ribero, Appert, etc.) contributed to this creation, using systems of axioms based on several different primitive notions such as derivation, neighborhood, surrounding, closed or open sets, closure or interior operators. The notion of the closure operator was also used in logics as early as in Tarski's 1929 paper [42] where he defines the consequence relation of a logical deductive system as a closure operator on an infinite set S satisfying a *finitary* axiom. Also observe that there are many notions equivalent to the notion of closure operator (see [36]) and in particular that the theory of closure systems is closely related to lattice theory since every (finite) lattice can be represented by a closure system. One can date back the notion of a Boolean (or truth) function to Boole (in his theory of *elective* functions). The definition of a Horn Boolean function as a Boolean function having a Horn (disjunctive normal) form appears for the first time (according to the authors) in Hammer and Kogan's 1992 paper [24]. However, the notion and name of the *Horn clause* come from the logician Alfred Horn who first pointed out the significance of such clauses in his 1951 paper "On sentences which are true of direct unions of algebras" [29]. This attribution is sometimes contested. For instance, Hodges [28] writes: "Horn clause logic is a part of first-order logic. It was first isolated by McKinsey [34]. The name 'Horn' is a historical accident. After McKinsey's paper in 1943, Alfred Tarski suggested investigating a more general class of sentences that are like Horn clauses except that they have arbitrarily many existential and universal quantifiers at the beginning. The sentences that Tarski described are now known as Horn sentences, because Tarski's colleague Alfred Horn [29] responded to Tarski's suggestion by showing that one of McKinsey's theorems is true for them too. This work of Horn is important in its own right, but it is not directly relevant to Horn clauses. (Henschen [26] p. 820 explains the name 'Horn clause' by a result of Horn [29] on Horn clauses; but the result is false, and it is not in [29])." On the other hand, Dechter and Pearl [15] write that the equivalence between Horn functions and families of subsets closed by intersection *appears to be a general folklore among many researchers, although we could not trace its precise origin*. However, in fact, Horn's 1951 paper [29] deals with Horn terms (i.e. propositional terms containing at most one complemented literal) and its Lemma 7 amounts to exactly saying that a Boolean function h is Horn (in the sense that it admits a Horn DNF) if and only if the family of its false points is closed by an intersection.

It is apparently Armstrong [3] who in the context of relational data bases has shown for the first time the one-to-one correspondence between the full family of functional dependencies (called here full implicational systems) and closure systems (Armstong called the closed sets saturated sets). However, one already finds a one-to-one correspondence between the so-called "transitive topologies" and the closure operators in Appert's paper quoted above [2]. The transitive topologies are nothing other than the binary relations between elements and subsets of a set which are the dual of the full unit implicational systems. These same correspondences have been rediscovered and/or generalized many times under various formulations. For instance, they appear in Buchi's book [9] where this author uses *dependence relations*, and in Doignon and Falmagne's book [17] between what they call *entailment relations* and the families of sets closed by unions (see also below).

One can ask what the link is between our implicational systems and logical systems ? First one can present the notions and results about implicational systems in the framework of propositional logic [20]. More deeply, Fagin displays an equivalence between the functional dependencies of relational databases (our implications) and the *implicational statements* of propositional logic [19]. An implicational statement of propositional logic is a conjunction of propositional (Boolean) variables implying a conjunction of propositional variables. Then, Fagin proves that a functional dependency is a consequence of a set of functional statements if and only if the corresponding implicational statement is a consequence of the corresponding set of implicational statements. On the other hand there are formal links between implicational systems and the ways to formalize the notion of logical consequence (see Scott 1974 [39] for an overview). As already mentioned, Hertz [27] (respectively, Tarski) used a binary relation between subsets and elements of a set of sentences (respectively, a closure operator) to formalize a notion of consequence. The connection between the two presentations is the same as the one used in this paper between an implicational system and a closure operator: $X \to y$ iff $y \in \varphi(X)$. Later, Gentzen in [22] introduced a relation where the right-hand side of the relation is a disjunction of sentences. Then in 1982 [40] Scott introduced the notion of *information systems* where there is an *entailment relation* between *consistent* subsets and elements of a set. Later, a one-to-one correspondence between Scott's information systems and *algebraic* $\cap$-*structures* has been displayed (see [14]). In the finite case, this correspondence is exactly the correspondence between the full implicational systems and the closure systems.

7. Conclusion

Since equivalent notions such as closure systems (or systems of sets closed by union), closure operators (or dual closure operators), full systems of implications (or of dependencies) and (pure) Horn functions have been studied by different authors in different domains (topology, lattice theory, hypergraph theory, choice functions, relational data bases, data mining and concept analysis, artificial intelligence and expert systems, knowledge spaces, logic and logic programming, theorem proving...), it is not surprising that one finds the same notions, results or algorithms under various names. For instance, in AI the meet-irreducible elements of a lattice of closed sets are called its characteristic sets, the associated closure operator is called the forward chaining procedure. In the context of Horn functions, a directed graph introduced at the earliest in 1987 by different authors on the set of the Boolean variables plays an important role. It can be shown that in the case of a pure Horn function, the relation defined by this graph is the inverse of the dependency relation defined in section 3.2 (it is also the domination relation defined in [4]). On the other side, on can also find many original results or algorithms, but which are generally known only in a specific domain. It would be very profitable to increase (or create) the communications between the various domains that use the same (or equivalent) notions and tools. Our paper is a first step in this direction and we intend to take further steps.

We also intend to work on the relationship between the canonical direct unit implicational basis, and the Duquenne–Guigues canonical basis mentioned in the introduction. Recall that this basis is an implicational system (IS for short) i.e. a binary relation on $\mathcal{P}(S)$ and that one can associate with it (as to any IS) an equivalent UIS by replacing each implication $A \to B$ by the set of implications $\{A \to b \ : \ b \in B\}$. We denote Σ_{can}, the UIS deduced of the Duquenne–Guigues basis by applying this rule. Consider in our example the two bases Σ_{can} (the UIS deduced from the canonical basis) and Σ_{cd} (the canonical direct unit basis):

$$\Sigma_{can} = \begin{cases} (1)\ 5 \to 4 & (2)\ 23 \to 4 \quad (3)\ 24 \to 3 \quad (4)\ 34 \to 2 \\ (5)\ 14 \to 2 & (6)\ 14 \to 3 \quad (7)\ 14 \to 5 \quad (8)\ 2345 \to 1 \end{cases}$$

$$\Sigma_{cd} = \begin{cases} (1)\,5 \to 4 & (2)\,23 \to 4 \quad (3)\,24 \to 3 \quad (4)\,34 \to 2 \\ (5)\,14 \to 2 & (6)\,14 \to 3 \quad (7)\,14 \to 5 \quad (8)\,25 \to 1 \\ (9)\,35 \to 1 & (10)\,15 \to 2 \quad (11)\,35 \to 2 \quad (12)\,15 \to 3 \\ (13)\,25 \to 3 & (14)\,123 \to 5 \end{cases}$$

Remark that Σ_{can} is a proper UIS since for every implication the conclusion is not included in the premise. Remark also that $\Sigma_{can} \not\subseteq \Sigma_{free}$ (Example 2) since the Σ_{can}-implication (8) does not belong to Σ_{free}. One can also verify that Σ_{can} is not direct, by considering the φ_Σ-closure of 15: $\pi_\Sigma(15) = 15 + 4$ by applying Σ_{can}-implication (1) and $\pi^2_\Sigma(15) = (15 + 4) + 2 + 3$ by applying Σ_{can}-implications (5) and (6). Therefore $\varphi_\Sigma(15) \neq \pi_\Sigma(15)$. We conclude that this example contradicts a conjecture of the literature (in [31]). Indeed, one observes that the premise of implication (10) of Σ_{cd} is not contained in a premise of any implication of Σ_{can}.

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References

[1] M.A. Aizerman, F.T. Aleskerov, Theory of Choice, Elsevier Science B.V, North Holland, 1995.
[2] A. Appert, Sur les topologies transitives, Bulletin de l'académie Royale de Belgique (Classe des Sciences) 23 (2) (1937) 135–142.
[3] W.W. Armstrong, Dependency structures of database relationships, Information Processing 74 (1974) 580–583.
[4] A. Berry, E. San Juan, Generalized domination in closure systems, Discrete Applied Mathematics 154 (7) (2006) 1064–1084.
[5] K. Bertet, Some algorithmical aspects using the canonical direct implicational basis, in: Fourth International Conference on Concept Lattice and their Applications, CLA'06, 2006, pp. 101–114.
[6] K. Bertet, M. Nebut, Efficient algorithms on the Moore family associated to an implicational system, DMTCS 6 (2) (2004) 315–338.
[7] G. Birkhoff, Lattice Theory, 1st ed., American Mathematical Society, 1940.
[8] G. Birkhoff, O. Frink, Representations of lattices by sets, Transactions of the American Mathematical Society 64 (1948) 299–316.
[9] J.R. Buchi, Finite automatas, their algebras and grammars, in: Toward a Theory of Formal Expressions, Springer Verlag, Berlin, 1989.
[10] N. Caspard, B. Leclerc, B. Monjardet, Ensembles Ordonnés Finis: Concepts, Résultats et Usages, vol. 60, Springer, 2007.
[11] N. Caspard, B. Monjardet, The lattice of closure systems, closure operators and implicational systems on a finite set: A survey, Discrete Applied Mathematics 127 (2) (2003) 241–269.
[12] E.F. Codd, A relational model of data for large shared data banks, Communications of the ACM 13 (6) (1970) 377–387.
[13] Y. Crama, P.L. Hammer, Boolean Functions, Cambridge University Press, 2009.
[14] B.A. Davey, H.A. Priestley, Introduction to Lattices and Orders, 2nd ed., Cambridge University Press, 1991.
[15] R. Dechter, J. Pearl, Structure identification in relational data, Artificial Intelligence 58 (1992) 237–270.
[16] J. Demetrovics, H. Nam-Son, Databases, closure operations and Sperner families, in: P. Frankl, et al. (Eds.), Extremal Problems for Finite Sets, János Bolyai Mathematical Society, New Haven, 1991, pp. 199–203.
[17] J.P. Doignon, J.C. Falmagne, Knowledge Spaces, Springer Verlag, Berlin, 1999.
[18] V. Duquenne, Latticial structures in data analysis, Theoretical Computer Science 217 (2) (1999) 407–436.
[19] R. Fagin, Functional dependencies in a relational database and propositional logic, IBM Journal of Research and Development 21 (6) (1977) 534–544.
[20] B. Ganter, Attribute exploration with background knowledge, Theoretical Computer Science 217 (1999) 215–233.
[21] B. Ganter, R. Wille, Formal Concept Analysis, Mathematical Foundations, Springer Verlag, Berlin, 1999.
[22] G. Gentzen, Investigations into logical deduction, in: M.E. Szabo (Ed.), The Collected Papers of Gerhard Gentzen, North-Holland, 1969.
[23] J.L. Guigues, V. Duquenne, Familles minimales d'implications informatives résultant d'un tableau de données binaires, Mathematiques & Sciences Humaines 95 (1986) 5–18.
[24] P.L. Hammer, A. Kogan, Horn's functions and their dnfs, Information Processing Letters 44 (1992) 23–29.
[25] D. Hand, H. Mannila, P. Smyth, Principles of Data-Mining, MIT Press, 2001.
[26] L.J. Henschen, Semantic resolution for Horn sets, IEEE Transactions on Computers 25 (1976) 816–822.
[27] P. Hertz, Über Axiomensysteme für beliebige Satzsysteme, Mathematische Annalen 101 (1927) 457–514.

[28] W. Hodges, Logical features of Horn clauses, in: In M. Gabbay, C.J. Hogger, J.A. Robinson, J. Siekmann (Eds.), in: Handbook of Logic in Artificial Intelligence and Logic Programming, Logical Foundations, vol. 1, 1993, pp. 449–503.
[29] A. Horn, On sentences which are true of direct unions of algebras, Journal of Symbolic Logic 16 (1951) 14–21.
[30] T. Ibaraki, A. Kogan, K. Makino, Functional dependencies in Horn theories, Artificial Intelligence 108 (1999) 1–30.
[31] K. Ben-Khalifa, S. Montameny, Horn-representation of a concept lattice, in: J. Diatta, P. Eklund, M. Liquière (Eds.), Fifth International Conference on Concept Lattices and their Applications, CLA'2007, Montpellier, France, October 24–26, 2007, pp. 181–191.
[32] D. Maier, The Theory of Relational Databases, Computer Sciences Press, 1983.
[33] H. Mannila, K.J. Räihä, The Design of Relational Databases, Addison-Wesley, 1992.
[34] J.C.C. McKinsey, The decision problem for some classes of sentences without quantifiers, Journal of Symbolic Logic 8 (1943) 61–76.
[35] B. Monjardet, Arrowian characterizations of latticial federation consensus functions, Mathematical Social Sciences 20 (1) (1990) 51–71.
[36] B. Monjardet, The presence of lattice theory in discrete problems of mathematical social sciences, Why? Mathematical Social Sciences 46 (2) (2003) 103–144.
[37] E.H. Moore, On a form of general analysis with applications to linear differential and integral equations, in: Atti del IV Congr. Int. dei Mat. II, Roma, 1909, pp. 98–114.
[38] A. Rusch, R. Wille, Knowledge spaces and formal concept analysis, in: H.H. Bock, W. Polasek (Eds.), Data Analysis and Information Systems, Springer Verlag, Berlin, 1995, pp. 427–436.
[39] D. Scott, Completeness and axiomatizability in many-valued logic, in: L. Henkin, et al. (Eds.), Proceedings of the Tarski Symposium, American Mathematical Society, 1974, pp. 411–435.
[40] D.S. Scott, Domains for denotational semantics, in: Automata, Languages and Programming, vol. 140, Berlin, 1982, pp. 577–613.
[41] R. Taouil, Y. Bastide, Computing proper implications, in: 9th International Conference on Conceptual Structures, Stanford, USA, 2002.
[42] A. Tarski, Remarques sur les notions fondamentales de la méthodologie des mathématiques, Annales de la Société Polonaise de Mathématiques 7 (1929) 270–272.
[43] M. Wild, A theory of finite closure spaces based on implications, Advances in Mathematics 108 (1) (1994) 118–139.